LECTURE 17 DERIVATIVE OF LOGARITHMIC FUNCTIONS

Last class, we derived the formula for the derivative of the inverse $f^{-1}(x)$ of a differentiable function $f(x)$, given that $f'(x) \neq 0$ everywhere in its domain. Now, we go visit some classical inverse pairs, such as the logarithmic and exponential function, trig and inverse trig.

We review some basic logarithmic operations. First, what really is logarithm? How did it come about? Now, everyone knows, 4 is the 2 to the power of 2, i.e. $4 = 2^2$, focus on the exponent. How about 5 then? What power of 2 gives 5, i.e. find x such that

$$2^x = 5.$$

Mathematicians then said, let's call it something, since we don't have an expression for it. Let's call $x = \log_2 5$, read as logarithmic of 5 base 2. It is a number such that when you raise 2 to this power, you get 5. In other words, we have the identity,

$$2^{\log_2 5} = 5.$$

Now, let's compile some known identities about $\log_a b$.

(1) (exponentiation trick) By definition,

$$a^{\log_a b} = b.$$

This implies that when you see a number b, you can rewrite it as $a^{\log_a b}$. We will see how this is useful later.

(2) (product rule for log)

$$\log_a (bc) = \log_a b + \log_a c.$$

Proof. Let consider two numbers $x = \log_a b$ and $y = \log_a c$. Then, $a^x = b$ and $a^y = c$. We further see that

$$a^x a^y = bc \implies a^{x+y} = bc \implies x + y = \log_a (bc).$$

But $x + y = RHS$. Done. $\square$

(3) (quotient rule for log) Similarly,

$$\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c, \quad c \neq 0.$$

(4) (bring out the power)

$$\log_a b^c = c \log_a b.$$

Proof. Clearly if $c = 0$, this is true. Assume $c \neq 0$, let $x = c \log_a b$. Then

$$\frac{x}{c} = \log_a b \implies a^{\frac{x}{c}} = b \implies a^x = b^c \implies x = \log_a b^c = LHS.$$

$\square$

(5) (change of base formula)

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Proof. Let $x = \log_a b$. Then $a^x = b$. At the same time, by itself,

$$a^x = c^{\log_c a^x} = c^{x \log_c a}$$

and

$$b = c^{\log_c b}.$$

This means $c^{x \log_c a} = c^{\log_c b}$. Since c^y is a one-to-one function, we must have

$$x \log_c a = \log_c b \implies x = \frac{\log_c b}{\log_c a}.$$

$\square$

Logarithm Function

We know that e^x and $\ln(x)$ are inverses of each other. In other words, $\ln(x)$ is ask for the value of the power we must raise on the number e to obtain x, but e raised to this power, that is, $e^{\ln(x)}$, should give you the value back x. Note that $\ln(x) > 0$ for $x > 0$, and is undefined (in real numbers) for $x \leq 0$.

As we see just now, the derivative of inverse function relates to the derivative of the original function. The derivative of the exponential function seems easy enough.

$$\frac{d}{dx}e^x = e^x.$$

Then, how about $\frac{d}{dx}\ln(x)$? Let $f(x) = e^x$ and we know $f^{-1}(x) = \ln(x)$. By the same formula, we find that

$$\frac{d}{dx}(\ln(x)) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

One can also do implicit differentiation to arrive at the same result. Consider $y = \ln(x)$ and we want $\frac{dy}{dx}$.

$$y = \ln(x) \implies e^y = x \implies \frac{d}{dx}e^y = \frac{d}{dx}(x)$$
$$\implies e^y\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

Furthermore, by chain rule, suppose we now have $f(x) = \ln(u(x))$ where u is a differentiable function ($u(x) > 0$). Then,

$$\begin{aligned}\frac{d}{dx}(\ln(u)) &= \frac{df}{du}\cdot\frac{du}{dx}\\ &= \frac{1}{u}\cdot u'\end{aligned}$$

Example. Find the derivative of $\ln(bx)$ for $b > 0$.

Solution.

$$\frac{d}{x}(\ln(bx)) = \frac{1}{bx}\frac{d}{dx}(bx) = \frac{1}{bx}\cdot b = \frac{1}{x}.$$

So your result is independent of b.

Example 1. Find the derivative of $\ln|x|$.

Solution. By the chain rule formula, note that $u(x) = |x|$, we must have

$$\begin{aligned}\frac{d}{dx}\ln|x| &= \frac{d}{du}\ln(u)\cdot\frac{du}{dx}\\ &= \frac{1}{u}\cdot\frac{x}{|x|}\\ &= \frac{1}{|x|}\cdot\frac{x}{|x|}\\ &= \frac{x}{x^2}\\ &= \frac{1}{x}.\end{aligned}$$

This means, $\frac{1}{x}$ is not only the derivative of $\ln(x)$ for $x > 0$, but also of $\ln(-x)$ for $x < 0$.